Finite speed of propagation in 1-D degenerate Keller-Segel system

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Abstract

We consider the following Keller-Segel system of degenerate type:

$$\text{(KS)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big(\dfrac{\partial u^m}{\partial x} - u^{q-1}\cdot\dfrac{\partial v}{\partial x}\Big), & x\in\mathbb{R},\ t>0,\\ 0 = \dfrac{\partial^2 v}{\partial x^2} - \gamma v + u, & x\in\mathbb{R},\ t>0,\\ u(x,0)=u_0(x), & x\in\mathbb{R}, \end{cases}$$

where $m>1,\ \gamma>0,\ q\geq 2m$. We shall first construct a weak solution $u(x,t)$ of (KS) such that u^{m-1} is Lipschitz continuous and such that $u^{m-1+\delta}$ for $\delta>0$ is of class C^1 with respect to the space variable x. As a by-product, we prove the property of finite speed of propagation of a weak solution $u(x,t)$ of (KS), *i.e.,* that a weak solution $u(x,t)$ of (KS) has a compact support in x for all $t>0$ if the initial data $u_0(x)$ has a compact support in $\mathbb{R}$. We also give both upper and lower bounds of the interface of the weak solution u of (KS).

1 Introduction

We consider the following Keller-Segel system of degenerate type:

$$\text{(KS)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big(\dfrac{\partial u^m}{\partial x} - u^{q-1}\cdot\dfrac{\partial v}{\partial x}\Big), & x\in\mathbb{R},\ t>0,\\ 0 = \dfrac{\partial^2 v}{\partial x^2} - \gamma v + u, & x\in\mathbb{R},\ t>0,\\ u(x,0)=u_0(x), & x\in\mathbb{R}, \end{cases}$$

where $m > 1,\ \gamma > 0,\ q \geq 2m$. The initial data u_0 is a non-negative function and in $L^1 \cap L^\infty(\mathbb{R})$ with $u_0^m \in H^1(\mathbb{R})$. This equation is often called as the Keller-Segel model describing the motion of the chemotaxis molds. (see *e.g.,* [5].)

The aim of this paper is to construct a weak solution $u(x,t)$ of (KS) such that u^{m-1} is Lipschitz continuous and such that $u^{m-1+\delta}$ for $\delta > 0$ is of class C^1 with respect to the space variable x. The regularity property whether u^{m-1} is Lipschitz continuous or of class C^1 plays an important role for the investigation of the behaviour of the interface to the solution u of (KS). Our result shows that the power $m-1$ to u exhibits the borderline behaviour between Lipschitz continuity and C^1-regularity. Indeed, as a by-product of Lipschitz continuity for u^{m-1}, we prove that a weak solution $u(x,t)$ of (KS) possesses the property of finite speed of propagation *i.e.,* that a weak solution $u(x,t)$ of (KS) has a compact support in x for all $t > 0$ if the initial data $u_0(x)$ has a compact support in $\mathbb{R}$.

Similar results have been obtained for the porous medium equation:

$$\text{(PME)} \qquad \begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U^m}{\partial x^2}, & x \in \mathbb{R},\ t > 0, \\ U(x,0) = U_0(x), & x \in \mathbb{R}. \end{cases}$$

It is known that the comparison principle gives both upper and lower bounds of all solutions U to (PME) by means of the *Barenblatt solution* V_B which is an exact solution of (PME). Hence the property of finite speed of propagation of U is a direct consequence of the explicit form of V_B since supp $V_B(\cdot,t)$ is compact in $\mathbb{R}$ for all time t.

Our purpose is to prove the property of finite speed of propagation for (KS) to which the comparison principle is not available. To this end, one makes use of the notion of the domain of dependence which is useful for the proof of uniqueness of solutions to the linear wave equations. For instance, the half-cone like region D_T defined by

$$D_T \ := \ \Big\{(x,t);\ -ct + a \leq x \leq ct + b, \quad 0 \leq t < T\Big\}, \quad a < b,\ c > 0$$

makes it possible to prove that the solution of the linear wave equation with the propagation speed c vanishes on D_T for the initial data u_0 such that $u_0(x) \equiv 0$ on $I \equiv [a,b]$.

To deal with (KS), we generalize such an idea, and consider the *curved* half-cone like region. Indeed, suppose that $u_0(x) = 0$ on I. Then our curved half-cone like region D_T with respect to I can be expressed by

$$\text{(1.1)} \qquad D_T \ := \ \Big\{(x,t);\ \xi(t) \leq x \leq \Xi(t), \quad 0 \leq t < T\Big\},$$

where $\xi(t)$ and $\Xi(t)$ are the solutions of the following initial value problems:

$$\text{(IE)} \qquad \begin{cases} \xi'(t) = -\frac{\partial}{\partial x}\Big(\frac{m}{m-1}u^{m-1}\Big)(\xi(t),t) + u^{q-2}\cdot\frac{\partial v}{\partial x}(\xi(t),t), & \xi(0) = a, \\ \Xi'(t) = -\frac{\partial}{\partial x}\Big(\frac{m}{m-1}u^{m-1}\Big)(\Xi(t),t) + u^{q-2}\cdot\frac{\partial v}{\partial x}(\Xi(t),t), & \Xi(0) = b. \end{cases}$$

Unfortunately, Lipschitz continuity of u^{m-1} is too weak to ensure the existence of solutions $\{\xi(t), \Xi(t)\}$ to (IE). Hence we need to regularize u by u_ε with small parameter $\varepsilon > 0$, and

deal with the approximating solutions $\{\xi_\varepsilon(t), \Xi_\varepsilon(t)\}$ which correspond to (IE) with u replaced by u_ε. It is shown that Lipschitz continuity of u^{m-1} guarantees the existence of uniform limit $\{\xi(t), \Xi(t)\}$ on $0 \le t \le T$ of $\{\xi_\varepsilon(t), \Xi_\varepsilon(t)\}$ as $\varepsilon \to 0$. Then we see that $u(x,t) = 0$ on D_T.

Our definition of a weak solution to (KS) now reads:

Definition 1 *Let m, γ and q be constants as $m > 1,\ \gamma > 0,\ q \ge 2$. Let u_0 be a non-negative function in $\mathbb{R}$ with $u_0 \in L^1 \cap L^\infty(\mathbb{R})$ and $u_0^m \in H^1(\mathbb{R})$. A pair of non-negative functions (u, v) defined in $\mathbb{R} \times [0, T)$ is said to be a weak solution of* (KS) *on $[0, T)$ if*

i) $u \in L^\infty(0, T; L^2(\mathbb{R}))$, $u^m \in L^2(0, T; H^1(\mathbb{R}))$,

ii) $v \in L^\infty(0, T; H^2(\mathbb{R}))$,

iii) *(u, v) satisfies (KS) in the sense of distributions: i.e.,*

$$\int_0^T \int_{\mathbf{R}} \left(\partial_x u^m \cdot \partial_x \varphi - u^{q-1} \partial_x v \cdot \partial_x \varphi - u \cdot \partial_t \varphi\right) \, dxdt \; = \; \int_{\mathbf{R}} u_0(x) \cdot \varphi(x, 0) \, dx,$$
$$\text{for all functions } \varphi \in C_0^\infty(\mathbb{R} \times [0, T)),$$

$$-\partial_x^2 v + \gamma v - u = 0 \qquad \text{for a.a. } (x, t) \text{ in } \mathbb{R} \times (0, T).$$

Concerning the local-in-time existence of weak solutions to (KS), the following result can be shown by a slight modification of argument developed by the author [15, Theorem 1.1].

Proposition 1.1 (local existence of weak solution and its L^∞ uniform bound) *Let $m > 1,\ \gamma > 0,\ q \ge 2$. Suppose that the initial data u_0 is non-negative everywhere. Then,* (KS) *has a non-negative weak solution (u, v) on $[0, T_0)$ with $T_0 = \Big(\|u_0\|_{L^\infty(\mathbf{R})} + 2\Big)^{-q}$. Moreover, $u(t)$ satisfies the following a priori estimate*

$$\|u(t)\|_{L^\infty(\mathbf{R})} \;\le\; \|u_0\|_{L^\infty(\mathbf{R})} + 2 \qquad \text{for all } t \in [0, T_0). \tag{1.2}$$

Remark 1. Concerning the global-in-time existence of weak solutions to (KS), the author and Kunii [17] obtained the following result: Let m, γ, q and the initial data u_0 be as in Definition 1. In the case $q < m + 2$, there exists a weak solution u of (KS) on $[0, \infty)$. On the other hand, in the case $q \ge m + 2$, the weak solution u of (KS) on $[0, \infty)$ can be constructed provided $\|u_0\|_{L^{\frac{q-m}{2}}(\mathbf{R})}$ is sufficiently small.

Now, we construct a weak solution u of (KS) with some additional regularity for the velocity potential u^{m-1}.

Theorem 1.2 *Let $m > 1, \gamma > 0$ and $q \geq 2m$. Let the initial data u_0 be as in Definition 1. In addition, we assume that u_0^{m-1} is Lipschitz continuous in $\mathbb{R}$. Then, the weak solution u of (KS) on $[0, T_0)$ given by Proposition 1.1 has the following additional properties (i) and (ii):*

(i) $u^{m-1}(x,t)$ is Lipschitz continuous with respect to x for all $0 \leq t < T_0$ with the estimate

$$\sup_{0<t<T_0} \|\partial_x u^{m-1}(t)\|_{L^\infty(\mathbb{R})} \ \leq \ C, \tag{1.3}$$

where $C = C(m, \gamma, q, u_0)$.

(ii) For every $\delta > 0$, $u^{m-1+\delta}(\cdot, t)$ is a C^1-function with respect to x for all $0 < t < T_0$ with the property that $\partial_x u^{m-1+\delta}(x,t) = 0$ at the points $(x,t) \in \mathbb{R} \times (0, T_0)$ such that $u(x,t) = 0$. Furthermore, in the case of $1 < m < 2$, we have $\partial_x u(x,t) = 0$ at the same points (x,t) as above.

Remark 2. (i) By the fundamental inequality

$$|u(x,t) - u(y,t)| \ \leq \ \begin{cases} \frac{2^{1/m-1}}{m-1} \|u\|_{L^\infty(Q_{T_0})}^{2-m} |u^{m-1}(x,t) - u^{m-1}(y,t)|, & 1 < m < 2, \\ |u^{m-1}(x,t) - u^{m-1}(y,t)|^{\frac{1}{m-1}}, & m \geq 2 \end{cases} \tag{1.4}$$

for all $x, y \in \mathbb{R},\ 0 < t < T_0$, we have by Theorem 1.2 that for every $0 < t < T_0$, $u(\cdot, t)$ is a Hölder continuous function in $\mathbb{R}$ with the exponent $\mu = \min\{1, \frac{1}{m-1}\}$.

(ii) For (PME), it is well-known that $\partial_x U^{m-1}(\cdot, t)$ becomes a discontinuous function in $\mathbb{R}$ after some definite time t. Our result in Theorem 1.2 makes it clear that continuity in x of $\partial_x u^p(x,t)$ is guaranteed for all $p > m-1$ and all $0 < t < T_0$. It seems to be an interesting question whether $\partial_x u^{m-1}(\cdot, t)$ is really discontinuous in $\mathbb{R}$ or not.

(iii) The hypothesis $q \geq 2m$ seems to be redundant. Indeed, such restriction on q stems from choice of the transformation ψ in (2.16) which may have a certain freedom to apply the Bernstein method to the uniform estimate of $\partial_x u^{m-1}$ for ε. It should be noted that in the case $\gamma = 0$, we can relax this restriction to $q \geq m+1$. See (2.26) below.

By (1.3) in Theorem 1.2, we can construct a pair of continuous functions $\xi(t)$ and $\Xi(t)$ on $[0, T_0)$ such that the region D_{T_0} defined by (1.1) belongs to the interior of the domain surrounded by the interface of u, which leads us to the property of finite speed of propagation to (KS).

Theorem 1.3 (property of finite propagation speed) *Let $m > 1,\ \gamma > 0$ and $q \geq 2m$. Let the initial data u_0 be as in Definition 1. In addition, we assume that $u_0(x) = 0$ on some interval $I \equiv [a, b]$ and that u_0^{m-1} is Lipschitz continuous in $\mathbb{R}$. Suppose that u is the weak solution of (KS) on $[0, T_0)$ given by Theorem 1.2. Then, there exists a pair $\{\xi(t), \Xi(t)\}$ of continuous functions on $[0, T_0)$ with the following properties (i) and (ii):*

(i) $\xi, \Xi \in W^{1,\infty}(0, T_0)$ with $\xi(0) = a,\ \Xi(0) = b$;

(ii) $u(x,t) = 0$ for $\xi(t) \leq x \leq \Xi(t),\ 0 \leq t < T_0$.

Remark 3. (i) Concerning (PME), the interface of U can be explicitly determined by the solutions $\hat{\xi}(t)$ and $\hat{\Xi}(t)$ of the following initial value problems:

$$\begin{cases} \hat{\xi}'(t) &= -\frac{\partial}{\partial x}\Big(\frac{m}{m-1}U^{m-1}\Big)(\hat{\xi}(t),t), \quad \hat{\xi}(0)=a,\\ \hat{\Xi}'(t) &= -\frac{\partial}{\partial x}\Big(\frac{m}{m-1}U^{m-1}\Big)(\hat{\Xi}(t),t), \quad \hat{\Xi}(0)=b. \end{cases}$$

Indeed, by the comparison principle Knerr [6] showed that if $U_0(x)=0$ on some interval $I=[a,b]$ and $U_0(x)>0$ on $I^c=\mathbb{R}\backslash I$, then it holds that $U(x,t)=0$ for $\hat{\xi}(t)\le x\le \hat{\Xi}(t)$ and $U(x,t)>0$ for $x<\hat{\xi}(t)$ and $x>\hat{\Xi}(t)$ for all $0\le t<\infty$. We call such $\hat{\xi}(t)$ and $\hat{\Xi}(t)$ the *interface* of (PME).

(ii) Compared with (PME), it is not clear whether (IE) determines the exact interface of (KS) to which the comparison principle is not available. However, if $\xi_1(t)$ and $\Xi_1(t)$ are the interface of (KS), *i.e.,* that $\xi_1(t)$ and $\Xi_1(t)$ have the property that

$$u(x,t)=0 \ \text{ in } I_t:=[\xi_1(t),\Xi_1(t)] \quad \text{and} \quad u(x,t)>0 \text{ in some neighbourhood outside of } I_t$$

for all $0\le t<T_0$, then we can see that $\xi(t)$ and $\Xi(t)$ given by Theorem 1.3 satisfy the estimates

$$\xi_1(t)\le\xi(t), \qquad \Xi(t)\le\Xi_1(t) \quad \text{for all } 0\le t<T_0.$$

Hence our result may be regarded as an estimate of the maximum and the minimum of the interface of (KS). Other observations were done by Mimura-Nagai [13] and Bonami-Hilhorst-Logak-Mimura [4].

This paper is organized as follows. In Section 2, we shall first recall the approximating problem $(\mathrm{KS})_\varepsilon$ of (KS) introduced by [17]. Our main purpose is devoted to the derivation of uniform gradient bound with respect to $\varepsilon>0$ of the approximating velocity potential $w_\varepsilon=\frac{m}{m-1}u_\varepsilon^{m-1}$, where u_ε is the smooth solution of $(\mathrm{KS})_\varepsilon$. Bernstein's method plays an important role to obtain our uniform estimate. (see *e.g.,* [12].) Then in Section 3, by the standard compactness argument, we shall prove the Lipschitz continuity of the velocity potential $w=\frac{m}{m-1}u^{m-1}$ for the weak solution u of (KS). It is expected that $\partial_x w(x,t)$ becomes a discontinuous function in x after some finite time t. However, we shall show that for $p>m-1$, $\partial_x u^p(x,t)$ is, in fact, a continuous function in $\mathbb{R}$ for all $t\in[0,T_0)$. Section 4 is devoted to the construction of continuous curves $\xi(t)$ and $\Xi(t)$ such that $u(x,t)=0$ on D_{T_0} defined by (1.1), which implies the property of the finite speed of propagation for (KS).

We will use the simplified notations:

1) $\partial_t=\frac{\partial}{\partial t}, \quad \partial_x=\frac{\partial}{\partial x}, \quad \partial_x^2=\frac{\partial^2}{\partial x^2}, \quad \partial_x^3=\frac{\partial^3}{\partial x^3},$
2) $\|\cdot\|_{L^r}=\|\cdot\|_{L^r(\mathbb{R})}, (1\le r\le\infty), \ \int\cdot\,dx:=\int_{\mathbb{R}}\cdot\,dx,$
3) $Q_T:=\mathbb{R}\times(0,T),$
4) When the weak derivatives $\partial_x u, \partial_x^2 u$ and $\partial_t u$ are in $L^p(Q_T)$ for some $p\ge1$, we say

that $u \in W_p^{2,1}(Q_T)$, *i.e.*,

$$W_p^{2,1}(Q_T) \;:=\; \Big\{u \in L^p(0,T;W^{2,p}(\mathbb{R})) \cap W^{1,p}(0,T;L^p(\mathbb{R}));$$
$$\|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|\partial_x u\|_{L^p(Q_T)} + \|\partial_x^2 u\|_{L^p(Q_T)} + \|\partial_t u\|_{L^p(Q_T)} < \infty\Big\}.$$

2 Approximating Problem

In order to justify the formal arguments, we introduce the following approximating equations of (KS):

$$(\mathrm{KS})_\varepsilon \begin{cases} \partial_t u_\varepsilon(x,t) = \partial_x\Big(\partial_x (u_\varepsilon+\varepsilon)^m - (u_\varepsilon+\varepsilon)^{q-2} u_\varepsilon \cdot \partial_x v_\varepsilon\Big), & (x,t) \in \mathbb{R}\times(0,T), \\ 0 = \partial_x^2 v_\varepsilon - \gamma v_\varepsilon + u_\varepsilon, & (x,t) \in \mathbb{R}\times(0,T), \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \mathbb{R}, \end{cases}$$

where $\varepsilon > 0$ is a positive parameter.

Let us introduce the following assumption on the initial data $u_{0\varepsilon}$ with $\varepsilon > 0$.

(A.1) $u_{0\varepsilon} \geq 0$ for all $x \in \mathbb{R}$ and $u_{0\varepsilon} \in W^{2,p}(\mathbb{R})$ with

$$\sup_{0<\varepsilon<1} \|u_{0\varepsilon}\|_{L^p(\mathbf{R})} \leq \|u_0\|_{L^p(\mathbf{R})} \qquad \text{for all } p \in [1,\infty],$$
$$\|u_{0\varepsilon} - u_0\|_{L^p(\mathbf{R})} \to 0 \quad \text{as } \varepsilon \to 0 \qquad \text{for all } p \in [1,\infty).$$

(A.2) $u_{0\varepsilon} \in W^{1,2}(\mathbb{R})$ with $\sup_{0<\varepsilon<1} \|\partial_x u_{0\varepsilon}\|_{L^2(\mathbf{R})} \leq \|\partial_x u_0\|_{L^2(\mathbf{R})}$.

Definition 2 We call $(u_\varepsilon, v_\varepsilon)$ a *strong solution* of $(\mathrm{KS})_\varepsilon$ if it belongs to $W_p^{2,1} \times W_p^{2,1}(Q_T)$ for some $p \geq 1$ and $(\mathrm{KS})_\varepsilon$ is satisfied almost everywhere.

For the strong solution, we consider the case $p = 3$ and introduce the space $\mathbf{W}(Q_T)$ defined by

$$\mathbf{W}(Q_T) \;:=\; W_3^{2,1} \times W_3^{2,1}(Q_T). \tag{2.1}$$

In [15]–[17], the following proposition concerning the existence of the strong solution was proved :

Proposition 2.1 *(local existence of approximating solution) Let $m \geq 1$, $\gamma > 0$, $q \geq 2$. We take $T_0 := (\|u_0\|_{L^\infty(\mathbf{R})} + 2)^{-q}$. Then, for every $\varepsilon > 0$ and every initial data $u_{0\varepsilon}$ satisfying the hypothesis* (A.1), $(\mathrm{KS})_\varepsilon$ *has a unique non-negative strong solution $(u_\varepsilon, v_\varepsilon)$ in $\mathbf{W}(Q_{T_0})$. Moreover, $u_\varepsilon(t)$ satisfies the following a priori estimate*

$$\|u_\varepsilon(t)\|_{L^\infty(\mathbf{R})} \;\leq\; \|u_0\|_{L^\infty(\mathbf{R})} + 2 \qquad \text{for all } t \in [0,T_0) \text{ and all } \varepsilon \in (0,1]. \tag{2.2}$$

Remark 4. (i) It should be noted that the time interval $[0, T_0)$ of the existence of the strong solution $(u_\varepsilon, v_\varepsilon)$ can be taken uniformly with respect to $\varepsilon > 0$.

(ii) The weak solution (u, v) of (KS) on $[0, T_0)$ given by Proposition 1.1 can be constructed as the weak limit of $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0$, where $(u_\varepsilon, v_\varepsilon)$ is the strong solution in Proposition 2.1. More precisely, by choosing a subsequence of $(u_\varepsilon, v_\varepsilon)$ which we denote by $(u_\varepsilon, v_\varepsilon)$ itself for simplicity, we have

$$\begin{array}{rcll} u_\varepsilon & \rightharpoonup & u & \text{weakly} - \text{star in } L^\infty(0, T_0; L^2(\mathbb{R})), \\ u_\varepsilon^m & \to & u^m & \text{weakly in } L^2(0, T_0; H^1(\mathbb{R})) \text{ and strongly in } C([0, T_0); L^2_{loc}(\mathbb{R})), \\ v_\varepsilon & \rightharpoonup & v & \text{weakly} - \text{star in } L^\infty(0, T_0; H^2(\mathbb{R})) \end{array}$$

as $\varepsilon \to 0$. In what follows, we assume that the sequence of approximating solutions $(u_\varepsilon, v_\varepsilon)$ satisfies the above convergence.

(iii) The strong solution $(u_\varepsilon, v_\varepsilon) \in \mathbf{W}(Q_{T_0})$ is more regular. Indeed, for every $\varepsilon > 0$, it can be shown that $u_\varepsilon, v_\varepsilon \in C^\infty(\mathbb{R} \times (0, T_0))$.

The following lemma gives the gradient estimate for the velocity potential u^{m-1}.

Lemma 2.2 *Let $m > 1$, $\gamma > 0$ and $q \geq 2m$. Let the initial data u_0 be as in Definition 1. For every $\varepsilon > 0$, we take $u_{0\varepsilon}$ so that the hypothesis* (A.1)–(A.2) *are satisfied. In addition, we assume that $u_{0\varepsilon}^{m-1}$ is Lipschitz continuous in $\mathbb{R}$. Then the strong solution u_ε of (KS)$_\varepsilon$ on $[0, T_0)$ given by Proposition 2.1 has the following property:*

$$\sup_{0<\varepsilon<1} \Big(\sup_{0<t<T_0} \|\partial_x (u_\varepsilon + \varepsilon)^{m-1}\|_{L^\infty(\mathbb{R})} \Big) \;\leq\; C, \tag{2.3}$$

where $C = C(m, \gamma, q, u_0)$.

Proof of Lemma 2.2. For the sake of simplicity, we denote $(u_\varepsilon, v_\varepsilon)$ by (u, v). To treat the velocity potential, let us define $w := \frac{m}{m-1}(u+\varepsilon)^{m-1}$. Multiplying the first equation of (KS)$_\varepsilon$ by $m(u+\varepsilon)^{m-2}$ and then rewriting the resultant identity in terms of w, we have

$$\begin{aligned} \partial_t w \;=\;& (m-1) w \cdot \partial_x^2 w + |\partial_x w|^2 - \Big((q-2)(u+\varepsilon)^{q-3} u + (u+\varepsilon)^{q-2} \Big) \cdot \partial_x v \cdot \partial_x w \\ & - (m-1)(u+\varepsilon)^{q-3} u \cdot \partial_x^2 v \cdot w. \end{aligned} \tag{2.4}$$

Now we apply Bernstein's method. Introducing the convex transformation $\psi : \bar{w} \to w$, determined below (2.16), we rewrite the identity (2.4) by means of $\bar{w} = \psi^{-1}(w)$ in the following form:

$$\begin{aligned} \partial_t \bar{w} \;=\;& (m-1)\psi \cdot \Big(\frac{\psi''}{\psi'} |\partial_x \bar{w}|^2 + \partial_x^2 \bar{w} \Big) + \psi' |\partial_x \bar{w}|^2 \\ & - \Big((q-2)(u+\varepsilon)^{q-3} u + (u+\varepsilon)^{q-2} \Big) \cdot \partial_x v \cdot \partial_x \bar{w} - (m-1) \frac{\psi}{\psi'} \cdot (u+\varepsilon)^{q-3} u \cdot \partial_x^2 v. \end{aligned} \tag{2.5}$$

We note that

$$\psi(\bar{w}) \;=\; w \;=\; \frac{m}{m-1}(u+\varepsilon)^{m-1}, \quad \psi^{'}(\bar{w})\cdot\partial_x\bar{w} \;=\; m(u+\varepsilon)^{m-2}\partial_x u, \tag{2.6}$$

$$(u+\varepsilon)^{q-i}\cdot\partial_x u \;=\; \frac{1}{m}\cdot\psi^{'}\cdot(u+\varepsilon)^{q-m-i+2}\cdot\partial_x\bar{w} \quad \text{for } i=3,4. \tag{2.7}$$

Differentiating both sides of (2.5) with respect to x, we obtain from (2.6) and (2.7) that

$$\begin{aligned}
\partial_t\partial_x\bar{w} \;=\;& (m-1)\psi'(\bar{w})\cdot\Big(\frac{\psi^{''}(\bar{w})}{\psi^{'}(\bar{w})}|\partial_x\bar{w}|^2+\partial_x^2\bar{w}\Big)\cdot\partial_x\bar{w}\\
&+\Big((m-1)\psi(\bar{w})\cdot\Big(\frac{\psi''(\bar{w})}{\psi'(\bar{w})}\Big)^{'}+\psi^{''}(\bar{w})\Big)\cdot(\partial_x\bar{w})^3\\
&+2\Big((m-1)\psi(\bar{w})\cdot\frac{\psi''(\bar{w})}{\psi'(\bar{w})}+\psi^{'}(\bar{w})\Big)\cdot\partial_x\bar{w}\cdot\partial_x^2\bar{w} \;+\; (m-1)\psi(\bar{w})\cdot\partial_x^3\bar{w}\\
&-(q-2)(q-3)\cdot\frac{1}{m}\cdot\psi^{'}(\bar{w})\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x v\cdot(\partial_x\bar{w})^2\\
&-2(q-2)\cdot\frac{1}{m}\cdot\psi^{'}(\bar{w})\cdot(u+\varepsilon)^{q-m-1}\cdot\partial_x v\cdot(\partial_x\bar{w})^2\\
&-(q-2)\cdot\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^2 v\cdot\partial_x\bar{w}\\
&-(q-2)\cdot\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x v\cdot\partial_x^2\bar{w}\\
&-\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-1}\cdot\partial_x^2 v\cdot\partial_x\bar{w}\\
&-\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-1}\cdot\partial_x v\cdot\partial_x^2\bar{w}\\
&-\frac{(m-1)^2}{m}\Big(\frac{\psi'(\bar{w})}{\psi(\bar{w})}\Big)^{'}\cdot\psi\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^2 v\cdot\partial_x\bar{w}\\
&-\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}\Big((q-2)u+\varepsilon\Big)\cdot\partial_x^2 v\cdot\partial_x\bar{w}\\
&-\frac{(m-1)^2}{m}\cdot\frac{\psi(\bar{w})^2}{\psi'(\bar{w})}\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^3 v.
\end{aligned} \tag{2.8}$$

Let us put $U := |\partial_x\bar{w}|^2$, then the following identities hold.

$$\partial_x\bar{w}\cdot\partial_x^2\bar{w}=\frac{1}{2}\partial_x U, \qquad \partial_x\bar{w}\cdot\partial_x^3\bar{w} \;=\; \frac{1}{2}\partial_x^2 U-(\partial_x^2\bar{w})^2. \tag{2.9}$$

Multiplying (2.8) by $\partial_x \bar{w}$ and using (2.9), the resultant equation in terms of U reads:

$$
\begin{aligned}
\frac{1}{2}\cdot\partial_t U \;=\;& \Big((m-1)\psi\big(\frac{\psi''}{\psi'}\big)' + m\psi''\Big)U^2 + \Big((m-1)\cdot\psi\cdot\frac{\psi''}{\psi'} + \frac{m+1}{2}\psi'\Big)\partial_x\bar{w}\cdot\partial_x U \\
& + (m-1)\psi\cdot\Big(\frac{1}{2}\partial_x^2 U - (\partial_x^2\bar{w})^2\Big) \\
& - (q-2)\cdot\frac{1}{m}\cdot\psi'(\bar{w})\cdot(u+\varepsilon)^{q-m-2}\cdot\Big((q-1)u+2\varepsilon\Big)\cdot\partial_x v\cdot\partial_x\bar{w}\cdot U \\
& - (q-2)\cdot\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^2 v\cdot U \\
& - (q-2)\cdot\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x v\cdot\frac{1}{2}\partial_x U \\
& \quad - \frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-1}\cdot\partial_x^2 v\cdot U \\
& \quad - \frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-1}\cdot\partial_x v\cdot\frac{1}{2}\partial_x U \\
& \quad - \frac{(m-1)^2}{m}\Big(\frac{\psi'(\bar{w})}{\psi(\bar{w})}\Big)'\cdot\psi\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^2 v\cdot U \\
& \quad - \frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}\Big((q-2)u+\varepsilon\Big)\cdot\partial_x^2 v\cdot U \\
& \quad - \frac{(m-1)^2}{m}\cdot\frac{\psi(\bar{w})^2}{\psi'(\bar{w})}\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^3 v\cdot\partial_x\bar{w}.
\end{aligned}
\tag{2.10}
$$

We consider a sequence $\{\eta_k(x)\}_{k=-\infty}^{\infty}$ of cut-off functions such that

$$\operatorname{supp}\eta_k = \{x\in\mathbb{R};\ -2+k\le x\le 2+k\}, \tag{2.11}$$

$$\eta_k(x) = 1 \qquad \text{for } -1+k\le x\le 1+k, \tag{2.12}$$

with

$$|\partial_x\eta_k(x)| \le c_1(\eta_k(x))^{\frac{3}{4}}, \quad -c_2\eta_k(x) \;\le\; \partial_x^2\eta_k(x) \;\le\; c_3 \qquad \text{for } x\in\mathbb{R}, \tag{2.13}$$

where c_1, c_2 and c_3 are positive constants independent of k. In Remark 5 below, we give an example of such $\{\eta_k\}_{k=-\infty}^{\infty}$.

Multiplication of (2.10) by η_k yields

$$
\begin{aligned}
\frac{1}{2}\cdot\partial_t(\eta_k U) \;=\;& \Big((m-1)\psi\big(\frac{\psi''}{\psi'}\big)' + m\psi''\Big)\eta_k U^2 \;-\; (m-1)\psi(\partial_x^2\bar{w})^2\eta_k \\
& + I\cdot\partial_x^2(\eta_k U) \;+\; J\cdot\partial_x(\eta_k U) \;+\; R_k,
\end{aligned}
\tag{2.14}
$$

where

$$
\begin{aligned}
I \;:=\;& \frac{m-1}{2}\psi, \\
J \;:=\;& \Big((m-1)\cdot\psi\cdot\frac{\psi''}{\psi'} + \frac{m+1}{2}\psi'\Big)\partial_x\bar{w} \;-\; (m-1)\psi\cdot\partial_x\eta_k \\
& - \frac{m-1}{2m}\cdot\psi(u+\varepsilon)^{q-m-2}\Big((q-1)u+\varepsilon\Big)\partial_x v,
\end{aligned}
$$

and R_k is regarded as the remainder term defined by

$$R_k \;:=\; \sum_{j=1}^{7} R_k^{(j)} \tag{2.15}$$

with

$$\begin{aligned}
R_k^{(1)} \;&:=\; -\Big((m-1)\cdot\psi\cdot\frac{\psi''}{\psi'}+\frac{m+1}{2}\psi'\Big)\partial_x\eta_k\cdot\partial_x\bar{w}\cdot U\\
&\quad -\frac{q-2}{m}\cdot\psi'(\bar{w})\cdot(u+\varepsilon)^{q-m-2}\cdot\Big((q-1)u+2\varepsilon\Big)\cdot\partial_x v\cdot\eta_k\cdot\partial_x\bar{w}\cdot U,\\
R_k^{(2)} \;&:=\; (m-1)\cdot\psi\cdot(\partial_x\eta_k)^2U,\\
R_k^{(3)} \;&:=\; \frac{m-1}{2m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}\Big((q-1)u+\varepsilon\Big)\cdot\partial_x v\cdot\partial_x\eta_k\cdot U,\\
R_k^{(4)} \;&:=\; -\frac{m-1}{m}\cdot\psi(\bar{w})\cdot(u+\varepsilon)^{q-m-2}\Big((2q-3)u+2\varepsilon\Big)\cdot\partial_x^2 v\cdot\eta_k U,\\
R_k^{(5)} \;&:=\; -\frac{m-1}{2}\cdot\psi\cdot\partial_x^2\eta_k\cdot U,\\
R_k^{(6)} \;&:=\; -\frac{(m-1)^2}{m}\cdot\Big(\frac{\psi'(\bar{w})}{\psi(\bar{w})}\Big)'\cdot\psi\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^2 v\cdot\eta_k U,\\
R_k^{(7)} \;&:=\; -\frac{(m-1)^2}{m}\cdot\frac{\psi(\bar{w})^2}{\psi'(\bar{w})}\cdot(u+\varepsilon)^{q-m-2}u\cdot\partial_x^3 v\cdot\partial_x\bar{w}\cdot\eta_k.
\end{aligned}$$

Now we choose the transformation $\psi(r)$ by

$$\psi(r) := \frac{m}{m-1}(\|u_0\|_{L^\infty(\mathbb{R})}+2+\varepsilon)^{m-1}\cdot\frac{r}{3}(4-r), \qquad 0\le r\le 1. \tag{2.16}$$

Then we observe that the coefficient of the first term $\eta_k U^2$ of the right-hand side in (2.14) is negative, in particular

$$\Big((m-1)\psi\big(\frac{\psi''}{\psi'}\big)'+m\psi''\Big)\eta_k U^2 \;\le\; -M\cdot\eta_k U^2 \qquad \text{for } (x,t)\in\mathbb{R}\times[0,T_0), \tag{2.17}$$

where $M := \dfrac{m(11m-3)}{12(m-1)} > 0$.

Indeed, since Proposition 2.1 states that

$$\frac{m}{m-1}\varepsilon^{m-1} \le w(x,t) \le \frac{m}{m-1}(\|u_0\|_{L^\infty(\mathbb{R})}+2+\varepsilon)^{m-1} \;=:\; L$$

holds for all $(x,t)\in\mathbb{R}\times[0,T_0)$, the definition $\psi(\bar{w})=w$ yields

$$(0<)\; 2-\sqrt{4-\frac{3m\varepsilon^{m-1}}{(m-1)L}} \le \bar{w}(x,t)\le 1, \qquad (x,t)\in\mathbb{R}\times[0,T_0) \tag{2.18}$$

for sufficiently small $\varepsilon > 0$. Moreover, by (2.18) we have

$$\frac{2L}{3} \;\le\; \psi'(\bar{w}) \;=\; \frac{2L}{3}(2-\bar{w}) \;\le\; \frac{4L}{3}, \quad \psi''(\bar{w}) \;=\; -\frac{2L}{3}, \tag{2.19}$$

$$\frac{1}{2} \;\le\; \left|\frac{\psi''}{\psi'}\right| \;\le\; 1, \quad -1 \;\le\; \Big(\frac{\psi''}{\psi'}\Big)' \;=\; \frac{\psi'\psi''' - (\psi'')^2}{(\psi')^2} \;=\; -\Big(\frac{\psi''}{\psi'}\Big)^2 \;\le\; -\frac{1}{4}. \tag{2.20}$$

Now from (2.19) and (2.20), we see that the left-hand side of (2.17) is bounded by

$$\begin{aligned}\Big((m-1)\psi\big(\frac{\psi''}{\psi'}\big)' + m\psi''\Big)\cdot \eta_k U^2 \;&\le\; \Big(-\frac{(m-1)L}{4} - \frac{2mL}{3}\Big)\cdot \eta_k U^2 \\ &=\; -\frac{m(11m-3)}{12(m-1)}\cdot \eta_k U^2 = -M\cdot \eta_k U^2 < 0.\end{aligned}$$

On the other hand, suppose that $\eta_k U$ attains its maximum at the point $(x_0, t_0) \in \mathbb{R} \times (0, T_0)$. Then it holds by $\eta_k \ge 0$ that

$$\frac{1}{2}\cdot \partial_t(\eta_k U)(x_0,t_0) \;\ge\; 0, \quad \partial_x^2(\eta_k U)(x_0,t_0) \;\le\; 0, \quad \partial_x(\eta_k U)(x_0,t_0) \;=\; 0. \tag{2.21}$$

Combining (2.17), (2.19)–(2.21) with (2.14), we obtain

$$M\cdot \eta_k U^2 \;\le\; \sum_{j=1}^{7} R_k^{(j)}, \tag{2.22}$$

where $\{R_k^{(j)}\}_{1\le j\le 7}$ is given by (2.15).

We are going to estimate the seven terms $\{R_k^{(j)}\}_{1\le j\le 7}$. To this end, firstly integrating the second equation of $(\mathrm{KS})_\varepsilon$ on $(-\infty, x)$, we have

$$\partial_x v(x,t) = \gamma \int_{-\infty}^{x} v(y,t)\, dy - \int_{-\infty}^{x} u(y,t)\, dy,$$

which yields

$$\sup_{0<t<T_0} \|\partial_x v(t)\|_{L^\infty(\mathbf{R})} \;\le\; 2\|u_0\|_{L^1(\mathbf{R})}. \tag{2.23}$$

Here we have used the fact that $v(x,t) > 0$ together with $\gamma \int_{-\infty}^{\infty} v(y,t)\, dy = \int_{-\infty}^{\infty} u(y,t)\, dy$ for all $t \in [0, T_0)$.

By (2.13), (2.23), Young's inequality, and the relation $\partial_x^2 v = \gamma v - u$, we have for $q \ge m+1$ that

$$\sum_{j=1}^{4} R_k^{(j)} \;\le\; C \;+\; \frac{M}{8}\cdot \eta_k U^2, \tag{2.24}$$

where C is a constant depending on m, γ, q and u_0. By (2.13) and Young's inequality, we have

$$R_k^{(5)} \;\leq\; \frac{c_2(m-1)}{2}\psi \cdot \eta_k U \;\leq\; C \;+\; \frac{M}{8} \cdot \eta_k U^2. \tag{2.25}$$

We are now going to estimate $R_k^{(6)}$. Since

$$-\frac{2L}{3} \cdot \frac{1}{\psi} - \Big(\frac{4L}{3}\Big)^2 \cdot \frac{1}{\psi^2} \;\leq\; \Big(\frac{\psi'}{\psi}\Big)' \;=\; \frac{\psi\psi'' - (\psi')^2}{(\psi)^2} \;\leq\; -\frac{2L}{3} \cdot \frac{1}{\psi} \;<\; 0,$$

and since $\partial_x^2 v = \gamma v - u$, by the hypothesis that $q \geq 2m$, we have that

$$\begin{aligned}
R_k^{(6)} \;&=\; -\frac{(m-1)^2}{m} \cdot \Big(\frac{\psi'(\bar{w})}{\psi(\bar{w})}\Big)' u \cdot \psi(\bar{w}) \cdot (u+\varepsilon)^{q-m-2} \cdot \partial_x^2 v \cdot \eta_k U \\
&\leq\; \frac{(m-1)^2}{m} \cdot \frac{2L}{3} \cdot (u+\varepsilon)^{q-m-1} \cdot \gamma v \cdot \eta_k U \\
&\quad + (m-1) \cdot \Big(\frac{4L}{3}\Big)^2 \cdot (u+\varepsilon)^{q-2m} \cdot \gamma v \cdot \eta_k U \\
&\leq\; C \;+\; \frac{M}{8} \cdot \eta_k U^2. && (2.26)
\end{aligned}$$

Indeed, since $v(x,t)$ satisfies

$$v(x,t) \;=\; \int_{\mathbf{R}} G(x-y) \cdot u(y,t)\; dy \tag{2.27}$$

with the Bessel potential $G(x)$ which can be express as

$$G(x) \;=\; \frac{1}{\sqrt{4\pi}} \int_0^\infty s^{-\frac{1}{2}} \cdot e^{-\gamma s - \frac{|x|^2}{4s}}\; ds \;=\; \frac{e^{-\sqrt{\gamma}|x|}}{2\gamma}, \tag{2.28}$$

it holds that $G \in L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$ and that

$$\sup_{0<t<T} \|v(t)\|_{L^\infty(\mathbf{R})} \;\leq\; C \sup_{0<t<T} \|u(t)\|_{L^1(\mathbf{R})} \;=\; C\|u_{0\varepsilon}\|_{L^1(\mathbf{R})} \;\leq\; \|u_0\|_{L^1(\mathbf{R})}.$$

By (2.6) and (2.7), it holds

$$\begin{aligned}
&\psi \cdot (u+\varepsilon)^{q-m-2} u \cdot \partial_x^3 v \\
&\quad=\; \frac{m}{m-1}(u+\varepsilon)^{q-3} u \cdot (\gamma \partial_x v - \partial_x u) \\
&\quad\leq\; \frac{m}{m-1} \cdot \gamma \cdot (u+\varepsilon)^{q-2} \cdot 2\|u_0\|_{L^1} + \frac{1}{m-1} \cdot \psi'(\bar{w})(u+\varepsilon)^{q-m-1} u \cdot \partial_x \bar{w}, && (2.29)
\end{aligned}$$

which yields

$$R_k^{(7)} \;\leq\; C \;+\; \frac{M}{8} \cdot \eta_k U^2. \tag{2.30}$$

Substituting (2.24), (2.25), (2.26) and (2.30) into (2.22), we obtain

$$M \cdot \eta_k U^2 \;\; \leq \;\; C.$$

Recalling $U = |\partial_x \bar{w}|^2$, we have by (2.11) and the above estimate that

$$|\partial_x \bar{w}|^2 \;=:\; U \;\; \leq \;\; C \qquad \text{for } -1+k \leq x \leq 1+k, \quad 0 \leq t < T_0, \tag{2.31}$$

where C is a constant independent of ε and k. Repeating the same argument as the above for $k = 0, \pm 1, \pm 2, \cdots$, we obtain the upper bound of $|\partial_x \bar{w}|$ which is independent of ε in the whole interval $\mathbb{R}$.

We recall the definition of w and $\psi(\bar{w})$:

$$\frac{m}{m-1}(u_\varepsilon + \varepsilon)^{m-1} \;\; = \;\; w \;\; = \;\; \psi(\bar{w}). \tag{2.32}$$

Differentiating both sides of (2.32) with respect to x, we have by (2.19) and (2.31) that

$$\left|\partial_x (u_\varepsilon + \varepsilon)^{m-1}\right| \;\; = \;\; \frac{m-1}{m} \cdot \psi^{'}(\bar{w}) \cdot |\partial_x \bar{w}| \;\; \leq \;\; C \quad \text{for all } (x,t) \in \mathbb{R} \times [0, T_0),$$

which yields (2.3). This proves Lemma 2.2.

Remark 5. In the proof of Lemma 2.2, we have used a sequence $\{\eta_k(x)\}_{k=-\infty}^{\infty}$ of cut-off functions with properties (2.11)–(2.13). Taking $\eta(x)$ as

$$\eta(x) = \begin{cases} 0 & \text{for} \quad x \leq -2 \\ 2(2+x)^4 & \text{for} \quad -2 < x < -\frac{3}{2}, \\ 1 - 2(x+1)^4 & \text{for} \quad -\frac{3}{2} < x \leq -1, \\ 1 & \text{for} \quad -1 \leq x \leq 1, \\ 1 - 2(x-1)^4 & \text{for} \quad 1 < x \leq \frac{3}{2}, \\ 2(2-x)^4 & \text{for} \quad \frac{3}{2} < x < 2, \\ 0 & \text{for} \quad x \geq 2, \end{cases}$$

and then defining η_k by $\eta_k(x) := \eta(x-k)$ for $k = 0, \pm 1, \pm 2, \cdots$, we see that $\{\eta_k(x)\}_{k=-\infty}^{\infty}$ has the desired properties (2.11)–(2.13).

3 Proof of Theorem 1.2

Let us first show that for every $t \in [0, T_0)$, $\{u_\varepsilon(\cdot, t)\}_{\varepsilon > 0}$ is a sequence of uniformly bounded and equi-continuous functions in $\mathbb{R}$. Indeed, the uniform bound is a consequence of (2.2). By (2.2), (2.3) and (1.4) with u replaced by $u_\varepsilon + \varepsilon$, it holds

$$|u_\varepsilon(x,t) - u_\varepsilon(y,t)| \;\; \leq \;\; C(\|u_0\|_{L^\infty} + 2)^2 |x-y|^\mu, \quad \mu = \min\{1, \frac{1}{m-1}\}$$

for all $x, y \in \mathbb{R}$, $0 \le t < T_0$, and all $\varepsilon > 0$, where C is the same constant as in (2.3). This implies that $\{u_\varepsilon(\cdot, t)\}_{\varepsilon>0}$ is a family of equi-continuous functions in $\mathbb{R}$ for all $0 \le t < T_0$. Hence by the Ascoli-Arzela theorem, there is a subsequence of $\{u_\varepsilon(\cdot, t)\}_{\varepsilon>0}$, which we denoted by $\{u_\varepsilon(\cdot, t)\}_{\varepsilon>0}$ itself such that

$$u_\varepsilon(\cdot, t) \quad \longrightarrow \quad u(\cdot, t) \qquad \text{as} \ \ \varepsilon \to 0 \tag{3.1}$$

uniformly in every compact interval $I \subset \mathbb{R}$.

On the other hand, by (2.3) and the weakly-star compactness of $L^\infty(Q_{T_0})$, there exists a sequence of $\{u_\varepsilon\}_{\varepsilon>0}$, which we denote by $\{u_\varepsilon\}_{\varepsilon>0}$ itself for simplicity, and a function $\tilde{u} \in L^\infty(Q_{T_0})$ such that

$$\partial_x(u_\varepsilon + \varepsilon)^{m-1} \ \to \tilde{u} \qquad \text{weakly} - \text{star} \quad \text{in } L^\infty(Q_{T_0})$$

with

$$\|\tilde{u}\|_{L^\infty(Q_{T_0})} \ \ \le \ \ \liminf_{\varepsilon \to +0} \|\partial_x(u_\varepsilon + \varepsilon)^{m-1}\|_{L^\infty(Q_{T_0})}.$$

By (3.1), it is easy to see that $\tilde{u} = \partial_x u^{m-1}$, which yields the desired estimate (1.3).

Next, we shall show that $\partial_x u^{m-1+\delta}(\cdot, t)$ is a continuous function in $\mathbb{R}$ for all $0 < t < T_0$ and for all $\delta > 0$ with the additional property that $\partial_x u^{m-1+\delta}(x, t) = 0$ at the point (x, t) such as $u(x, t) = 0$. To this aim, we follow a similar argument employed in Aronson [3]. Let $u(x_0, t_0) > 0$. Then we see by the standard argument that both $\partial_x u$ and $\partial_x u^{m-1+\delta}$ with $\delta > 0$ are continuous functions in a neighbourhood of (x_0, t_0). Therefore, it suffices to prove that $\partial_x u^{m-1+\delta}(\cdot, t)$ is a continuous function in a neighbourhood of x_1 such as $u(x_1, t) = 0$ with the additional property that $\partial_x u^{m-1+\delta}(x_1, t) = 0$. By virtue of (3.1), for every $t \in [0, T_0)$ and every compact interval $I \subset \mathbb{R}$, it holds that $u_\varepsilon(\cdot, t) \to u(\cdot, t)$ uniformly on I. Therefore, by Remark 2, there exists $a_0 > 0$ such that

$$\begin{aligned} 0 \le u_\varepsilon(x, t) \ &\le \ |u_\varepsilon(x, t) - u(x, t)| + |u(x, t) - u(x_1, t)| + u(x_1, t) \\ &\le \ 2a^\mu \end{aligned} \tag{3.2}$$

holds for all $x \in I_a(x_1) := \{x \in \mathbb{R}; |x - x_1| < a\}$ and for all $0 < a \le a_0$ and for all $0 < \varepsilon < 1$, where $\mu := \min\{1, \frac{1}{m-1}\}$.

On the other hand, since we have

$$u_\varepsilon^{m-1+\delta}(x, t) - u_\varepsilon^{m-1+\delta}(x', t) \ \ = \ \ \frac{m-1+\delta}{m-1} \int_{x'}^{x} u_\varepsilon^\delta(x, t) \cdot \partial_x u_\varepsilon^{m-1}(x, t) \ dx, \tag{3.3}$$

it follows from (3.2),(3.3) and Lemma 2.2 that

$$|u_\varepsilon^{m-1+\delta}(x, t) - u_\varepsilon^{m-1+\delta}(x', t)| \ \ \le \ \ C(2a^\mu)^\delta |x - x'| \qquad \text{for all } x, x' \in I_a(x_1) \tag{3.4}$$

and for all $0 < a \le a_0$ and for all $0 < \varepsilon < 1$, where C depends on m, γ, q, u_0 but not on ε. Letting $\varepsilon \to +0$ in (3.4), we have by (3.1) that

$$\begin{aligned} &|u^{m-1+\delta}(x, t) - u^{m-1+\delta}(x', t)| \\ &\quad \le \ C(2a^\mu)^\delta |x - x'| \quad \text{for all } x, x' \in I_a(x_1) \text{ and all } 0 < a \le a_0. \end{aligned} \tag{3.5}$$

Taking $x = x_1$ in (3.5) and then letting $x' \to x_1$, we have

$$|\partial_x u^{m-1+\delta}(x_1,t)| \ \leq \ C(2a^\mu)^\delta, \quad 0 < a \leq a_0.$$

Hence we have by letting $a \to 0$ that

$$\partial_x u^{m-1+\delta}(x_1,t) = 0.$$

Similarly, letting $x' \to x$ in (3.5), we have

$$|\partial_x u^{m-1+\delta}(x,t)| \ \leq \ C(2a^\mu)^\delta \qquad \text{for all } 0 < a \leq a_0,$$

which implies that $\partial_x u^{m-1+\delta}(\cdot,t)$ is continuous at x_1. Since x_1 can be taken arbitrary in such a way that $u(x_1,t) = 0$, we conclude that $\partial_x u^{m-1+\delta}(\cdot,t)$ is a continuous function in $\mathbb{R}$ for all $t \in [0,T_0)$ with the additional property that $\partial_x u^{m-1+\delta}(x,t) = 0$ for the point (x,t) such as $u(x,t) = 0$.

The case of $1 < m < 2$ can be handled in a similar manner as above and we conclude that $\partial_x u(\cdot,t)$ is a continuous function in $\mathbb{R}$ for all $t \in [0,T_0)$ with the additional property that $\partial_x u(x,t) = 0$ for the point (x,t) such as $u(x,t) = 0$. This completes the proof of Theorem 1.2.

4 Proof of Theorem 1.3

Let $(u_\varepsilon, v_\varepsilon)$ be the unique strong solution of $(\mathrm{KS})_\varepsilon$ given by Proposition 2.1. For a fixed $R > 0$, we take $a, b > 0$ such as $-R < a < b < R$ and consider the following ordinary differential equations:

$$(\mathrm{IE})_\xi : \begin{cases} \xi_\varepsilon'(t) = \frac{m}{m-1}\partial_x\big(u_\varepsilon+\varepsilon\big)^{m-1}(\xi_\varepsilon(t),t) - \big(u_\varepsilon+\varepsilon\big)^{q-3}u_\varepsilon \cdot \partial_x v_\varepsilon(\xi_\varepsilon(t),t), \quad 0 \leq t < T_0, \\ \xi_\varepsilon(0) = a, \end{cases}$$

and

$$(\mathrm{IE})_\Xi : \begin{cases} \Xi_\varepsilon'(t) = \frac{m}{m-1}\partial_x\big(u_\varepsilon+\varepsilon\big)^{m-1}(\Xi_\varepsilon(t),t) - \big(u_\varepsilon+\varepsilon\big)^{q-3}u_\varepsilon \cdot \partial_x v_\varepsilon(\Xi_\varepsilon(t),t), \quad 0 \leq t < T_0, \\ \Xi_\varepsilon(0) = b. \end{cases}$$

By Remark 4 (iii), (2.2), (2.3) and (2.23), we have

$$\partial_x(u_\varepsilon+\varepsilon)^{m-1} \in C^1([-2R,2R]\times[0,T_0)) \tag{4.1}$$

and

$$\sup_{0<\varepsilon<1}\Big(\sup_{0<t<T_0}\{\|\partial_x(u_\varepsilon+\varepsilon)^{m-1}(\cdot,t)\|_{L^\infty(-2R,2R)} + \|\big(u_\varepsilon+\varepsilon\big)^{q-3}u_\varepsilon\cdot\partial_x v_\varepsilon(\cdot,t)\|_{L^\infty(-2R,2R)}\}\Big)$$
$$\leq C,$$

where $C = C(m,\gamma,q,u_0)$. We now chose $R > 0$ large enough such that $\frac{2R}{C} > T_0$. Then, it follows from the well-known theorem on the existence and uniqueness of local solutions to

the initial value problem for the ordinary differential equations that both $(\mathrm{IE})_\xi$ and $(\mathrm{IE})_\Xi$ have a unique C^1-solution $\xi_\varepsilon(t)$ and $\Xi_\varepsilon(t)$ on $[0, T_0)$ for all $\varepsilon > 0$, respectively.

We consider the following domain:

$$D_\tau \ := \ \bigcup_{t\in[0,\tau]} I_t \times \{t\}, \quad I_t \ := \ \Big\{x \in \mathbb{R};\ \xi_\varepsilon(t) \le x \le \Xi_\varepsilon(t)\Big\} \quad \text{for } 0 < \tau < T_0.$$

By the local uniqueness of the initial value problem $(\mathrm{IE})_\xi$ and $(\mathrm{IE})_\Xi$, we obtain that

$$\xi_\varepsilon(t) < \Xi_\varepsilon(t) \qquad \text{for all } 0 \le t < T_0.$$

Let us define the gradient $\overrightarrow{\nabla}$ and the vector $\mathbf{F}$ on (x,t) by

$$\overrightarrow{\nabla} \ := \ (\partial_x, \partial_t), \quad \mathbf{F}(x,t) \ := \ \Big(-\partial_x(u_\varepsilon+\varepsilon)^m + (u_\varepsilon+\varepsilon)^{q-2}u_\varepsilon \cdot \partial_x v_\varepsilon, \ \ u_\varepsilon + \varepsilon \Big).$$

Then it follows from the first equation of $(\mathrm{KS})_\varepsilon$ that

$$\begin{aligned} &\int_{D_\tau} \overrightarrow{\nabla} \cdot \mathbf{F}(x,t)\ dxdt \\ &= \int_{D_\tau} \partial_t u_\varepsilon - \partial_x\Big(\partial_x(u_\varepsilon+\varepsilon)^m - (u_\varepsilon+\varepsilon)^{q-2}u_\varepsilon \cdot \partial_x v_\varepsilon\Big)\ dxdt \ = \ 0 \end{aligned} \tag{4.2}$$

for all $0 < \tau < T_0$. Taking two curves C_1 and C_2 as

$$C_1 := \{(x,t) = (\xi_\varepsilon(t), t);\ 0 < t < \tau\}, \quad C_2 := \{(x,t) = (\Xi_\varepsilon(t), t);\ 0 < t < \tau\},$$

we have

$$\partial D_\tau \ = \ I_0 \ \cup \ C_1 \ \cup \ C_2 \ \cup \ I_\tau.$$

Hence, the Stokes formula gives

$$\begin{aligned} 0 &= \int_{D_\tau} \overrightarrow{\nabla} \cdot \mathbf{F}(x,t)\ dxdt \\ &= \int_{\partial D_\tau} \mathbf{F}(x,t) \cdot \mathbf{n}\ dS \\ &= \int_a^b \mathbf{F}(x,0)\cdot(0,-1)\ dx - \int_{\Xi_\varepsilon(\tau)}^{\xi_\varepsilon(\tau)} \mathbf{F}(x,t)\cdot(0,1)\ dx + \int_{C_1} \mathbf{F}\cdot\mathbf{n_1}\ dS + \int_{C_2} \mathbf{F}\cdot\mathbf{n_2}\ dS \\ &= -\int_a^b (u_{0\varepsilon}+\varepsilon)\ dx + \int_{\xi_\varepsilon(\tau)}^{\Xi_\varepsilon(\tau)} (u_\varepsilon+\varepsilon)\ dx + \int_{C_1} \mathbf{F}\cdot\mathbf{n_1}\ dS + \int_{C_2} \mathbf{F}\cdot\mathbf{n_2}\ dS, \end{aligned} \tag{4.3}$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ denote the unit outer normals to C_1 and C_2, respectively. Since

$$\mathbf{n_1} \ = \ \frac{(1, \xi_\varepsilon'(t))}{\sqrt{1+(\xi_\varepsilon'(t))^2}}, \qquad \mathbf{n_2} \ = \ \frac{(1, \Xi_\varepsilon'(t))}{\sqrt{1+(\Xi_\varepsilon'(t))^2}},$$

we have by $(\mathrm{IE})_\xi$ and $(\mathrm{IE})_\Xi$ that

$$\mathbf{F}\cdot\mathbf{n_1} \;=\; 0 \quad \text{on} \;\; C_1, \qquad \mathbf{F}\cdot\mathbf{n_2} \;=\; 0 \quad \text{on} \;\; C_2. \tag{4.4}$$

Combining (4.2)–(4.4), we have

$$\int_{\xi_\varepsilon(\tau)}^{\Xi_\varepsilon(\tau)} (u_\varepsilon(x,\tau)+\varepsilon)\;dx \;=\; \int_a^b (u_{0\varepsilon}(x)+\varepsilon)\;dx, \qquad 0\le \tau < T_0. \tag{4.5}$$

On the other hand, we obtain from (2.23), Proposition 2.1 and Lemma 2.2 that

$$\begin{aligned}
\sup_{0<\varepsilon<1} \|\xi_\varepsilon\|_{L^\infty(0,T_0)} &\le a + \Big(\frac{m}{m-1}\cdot C + 2(\|u_0\|_{L^\infty}+2)^{q-2}\cdot\|u_0\|_{L^1}\Big)\cdot T_0,\\
\sup_{0<\varepsilon<1} \|\xi_\varepsilon'\|_{L^\infty(0,T_0)} &\le \frac{m}{m-1}\cdot C + 2(\|u_0\|_{L^\infty}+2)^{q-2}\cdot\|u_0\|_{L^1}.
\end{aligned}$$

Hence it follows by the Ascoli-Arzela theorem that there exists a subsequence of $\{\xi_\varepsilon(t)\}$, still denoted by $\{\xi_\varepsilon(t)\}_{\varepsilon>0}$, and a function $\xi\in C^{0,1}[0,T_0)$ such that

$$\xi_\varepsilon(t) \;\to\; \xi(t) \qquad \text{as } \varepsilon\to 0 \quad \text{uniformly} \quad \text{for every} \;\; t\in[0,T_0). \tag{4.6}$$

Obviously, a similar argument to $\Xi_\varepsilon(t)$ also holds, and there exist a subsequence of $\{\Xi_\varepsilon(t)\}_{\varepsilon>0}$, still denoted by $\{\Xi_\varepsilon(t)\}$, and $\Xi\in C^{0,1}[0,T_0)$ such that

$$\Xi_\varepsilon(t) \;\to\; \Xi(t) \qquad \text{as } \varepsilon\to 0 \quad \text{uniformly} \quad \text{for every} \;\; t\in[0,T_0). \tag{4.7}$$

Since $u_0\equiv 0$ on $[a,b]$, by letting $\varepsilon\to 0$ in (4.5), we have

$$\int_{\xi(t)}^{\Xi(t)} u(x,t)\;dx \;=\; \int_a^b u_0(x)\;dx \;=\; 0 \tag{4.8}$$

for all $0\le t<T_0$. Indeed, we may assume

$$-2R < \xi_\varepsilon(t) < \Xi_\varepsilon(t) < 2R \qquad \text{for all } \varepsilon>0, \text{ and all } 0\le t<T_0,$$

where $R>0$ is the same as in (4.1). Hence it follows from (3.1), (4.6), (4.7) and Proposition 1.1 that

$$\begin{aligned}
&\Big|\int_{\xi_\varepsilon(t)}^{\Xi_\varepsilon(t)} (u_\varepsilon+\varepsilon)\;dx - \int_{\xi(t)}^{\Xi(t)} u\;dx\Big|\\
&\le \Big|\int_{\xi_\varepsilon(t)}^{\Xi_\varepsilon(t)} (u_\varepsilon-u)\;dx\Big| \;+\; \varepsilon\int_{\xi_\varepsilon(t)}^{\Xi_\varepsilon(t)} dx \;+\; \Big|\int_{\xi_\varepsilon(t)}^{\Xi_\varepsilon(t)} u\;dx - \int_{\xi(t)}^{\Xi(t)} u\;dx\Big|\\
&\le \Big(\sup_{-2R\le x\le 2R}|u_\varepsilon(x,t)-u(x,t)|+\varepsilon\Big)(\Xi_\varepsilon(t)-\xi_\varepsilon(t))\\
&\qquad + \|u\|_{L^\infty(Q_{T_0})}\Big(|\Xi_\varepsilon(t)-\Xi(t)|+|\xi_\varepsilon(t)-\xi(t)|\Big)\\
&\le 4R\Big(\sup_{-2R\le x\le 2R}|u_\varepsilon(x,t)-u(x,t)|+\varepsilon\Big) \;+\; \Big(\|u_0\|_{L^\infty}+2\Big)\Big(|\Xi_\varepsilon(t)-\Xi(t)|+|\xi_\varepsilon(t)-\xi(t)|\Big)\\
&\to 0 \qquad \text{as } \varepsilon\to +0,
\end{aligned}$$

which yields (4.8). Since u is non-negative in $\mathbb{R} \times [0, T_0)$, we conclude from (4.8) that

$$u(x,t) = 0 \qquad \text{for} \quad \xi(t) \leq x \leq \Xi(t),\ 0 \leq t < T_0.$$

This proves Theorem 1.3.

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